MTH 520/622: Introduction to hyperbolic geometry

Homework I

(Due 25/08)

1. Consider the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

   (a) Show that the sequence $a_n = \{1/n\}$ converges to 0, and the sequence $b_n = \{n\}$ converges to ∞, in $\bar{\mathbb{C}}$.

   (b) Show that if K is a closed and bounded subset of $\mathbb{C}$, then $(\mathbb{C} \setminus K) \cup \{\infty\}$ is open in $\bar{\mathbb{C}}$.

   (c) Show that every open subset of $\bar{\mathbb{C}}$ is either an open subset of $\mathbb{C}$ or is the complement in $\bar{\mathbb{C}}$ of a closed and bounded subset of $\mathbb{C}$.

2. Consider the stereographic projection $p : S^2 \setminus \{N\} \to \mathbb{C}$, where N denotes the north pole of S^2.

   (a) Derive explicit expressions for p and p^{-1}.

   (b) Show that p is a homeomorphism.

   (c) Consider the extension $\bar{p} : S^2 \to \bar{\mathbb{C}}$ of p to S^2 defined by

$$\bar{p}(x) = \begin{cases} p(x), & \text{if } x \in S^2 \setminus \{N\}, \text{ and} \\ \infty, & \text{if } x = N. \end{cases}$$

   Show that $\bar{p}$ is a homeomorphism.

3. Consider the map $r : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ idefined by

$$r(z) = \begin{cases} 1/z, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = 0, \text{ and} \\ 0, & \text{if } z = \infty. \end{cases}$$

   (a) Show that r is a homeomorphism.

   (b) Describe geometrically the map $(\bar{p})^{-1} \circ r : S^2 \to S^2$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial such that $\deg(f) \geq 1$, and let $g : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be the extension of p defined by

$$g(z) = f(z), \text{ for } z \in \mathbb{C} \text{ and } g(\infty) = \infty.$$

   (a) Show that g is continuous in $\bar{\mathbb{C}}$.

   (b) Show that $g \in \text{Homeo}(\bar{\mathbb{C}})$ iff $\deg(f) = 1$.